Assignment 6

This homework is due Friday March 11.

There are total 50 points in this assignment. 45 points is considered 100%. If you go over 45 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should exhibit your work and contain full proofs. Bare answers will not earn you much.

This assignment covers Sections 3.3, 4.1, 4.3, and part of 4.4 of Textbook.

(1) [3pt] Establish which of the following functions are harmonic.
   (a) $u(x,y) = e^x \cos y$.
   (b) $u(x,y) = \arctan \frac{y}{x}$, $x \neq 0$.
   (c) $u(x,y) = x^2 + y^2$.
   (d) $u(x,y) = \ln(x^2 + y^2)$, $(x,y) \neq (0,0)$.

(2) [7pt] For a given $u(x,y)$ find $v(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$ is analytic. If no such v exists, explain why.

   (a) $u(x,y) = x^3 - y^3$.
   (b) $u(x,y) = 4x^3y - 4xy^3$.
   (c) $u(x,y) = \sin y \sinh x$.
   (d) $u(x,y) = e^y \sin x$.

(3) [5pt] Find the following limits. (*Hint* for (b): $\frac{n+i^n}{n} = \frac{1+i^n/n}{1}$. In (c), (d) similarly divide by the largest term in the numerator and the denominator.)

   (a) $\lim\limits_{n\to\infty} \left(\frac{1}{2} + \frac{i}{5}\right)^n$.
   (b) $\lim\limits_{n\to\infty} \frac{n+i^n}{n}$.
   (c) $\lim\limits_{n\to\infty} \frac{n^2+i2^n}{2^n}$.
   (d) $\lim\limits_{n\to\infty} \frac{(n+i)(1+ni)}{n^2}$.

(4) [5pt]
   (a) Show that $\sum\limits_{n=0}^{\infty} \left(\frac{1}{n+1+i} - \frac{1}{n+i}\right) = i$. (*Hint:* Look at partial sums.)
   (b) Does $\sum\limits_{n=1}^{\infty} \frac{i^n}{n}$ converge? Why? (*Hint:* Look at Re and Im of partial sums.)

(5) [5pt] Use the ratio test to establish whether the following series converge.
   (a) $\sum\limits_{n=0}^{\infty} \frac{(1+i)^n}{n2^n}$.
   (b) $\sum\limits_{n=1}^{\infty} \frac{(3+4i)^n}{n!}$.
   (c) $\sum\limits_{n=0}^{\infty} \frac{n!}{5^n(n^{10}+n+1)}$.

(6) [15pt] Find radius of convergence of the following series (use ratio or root formula, whichever you find better suited in each case).

   (a) $\sum\limits_{n=0}^{\infty} (-1+i)^n (z-1)^n$.
   (b) $\sum\limits_{n=0}^{\infty} (-1+i)^n z^{2n}$.
   (c) $\sum\limits_{n=0}^{\infty} \frac{z^n}{(3-4i)^n}$.
   (d) $\sum\limits_{n=0}^{\infty} \frac{(z+i)^n}{(3-4i)^n}$.
   (e) $\sum\limits_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!}$.
   (f) $\sum\limits_{n=0}^{\infty} n! z^n$.
   (g) $\sum\limits_{n=0}^{\infty} n! z^{n!}$.
   (h) $\sum\limits_{n=0}^{\infty} \left(\frac{4n^2}{2n+1} - \frac{6n^2}{3n+4}\right) z^n$.
   (i) $\sum\limits_{n=0}^{\infty} (2-(-1)^n)^n z^n$.
   (j) $\sum\limits_{n=0}^{\infty} \frac{n(n-1)z^n}{(3+4i)^n}$.

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(7) [3pt] Show that for $|z-i|<\sqrt{2}$, $\frac{1}{1-z}=\sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}}$.

(*Hint:* $\frac{1}{1-z}=\frac{1}{(1-i)-(z-i)}=\frac{1}{1-i}\left(\frac{1}{1-\frac{z-i}{1-i}}\right)$. Consider a geometric series with ratio $r=\frac{z-i}{1-i}$. In particular, when is $|r|<1$?)

(8) [7pt]

(a) Differentiate termwise the equality $\sum_{n=0}^{\infty} z^n=\frac{1}{1-z}$ twice.

(b) Show that $\sum_{n=0}^{\infty}(n+1)^2 z^n=\frac{1+z}{(1-z)^3}$. For what values of z is this equality valid? (*Hint:* Combine the series from (a) and its derivatives to get the coefficient $(n+1)^2$.)